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ORIGINAL PAPER

Borsuk–Ulam Property and Sectional Category

Cesar Augusto Ipanaque Zapata[1] · **Daciberg Lima Gonçalves**[1]

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Abstract

For a Hausdorff space X, a free involution $\tau : X \to X$ and a Hausdorff space Y, we discover a connection between the sectional category of the double covers $q : X \to X/\tau$ and $q^Y : F(Y, 2) \to D(Y, 2)$ from the ordered configuration space $F(Y, 2)$ to its unordered quotient $D(Y, 2) = F(Y, 2)/\Sigma_2$, and the Borsuk–Ulam property (BUP) for the triple $((X, \tau); Y)$. Explicitly, we demonstrate that the triple $((X, \tau); Y)$ satisfies the BUP if the sectional category of q is bigger than the sectional category of q^Y. This property connects a standard problem in Borsuk–Ulam theory to current research trends in sectional category. As an application of our results, we present a new lower bound for the index in terms of sectional category. We present several examples for whom the lower bound coincides with sectional category minus 1. We conjecture that the index of (M, τ) coincides with the sectional category of the quotient map $q : M \to M/\tau$ minus 1 for any CW complex M.

Keywords Borsuk–Ulam theorem · Sectional category · L–S category · Configuration spaces · Classifying maps

Mathematics Subject Classification 55M20 · 55M30 · 57M10 · 55r80 · 55R35.

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✉ Cesar Augusto Ipanaque Zapata
cesarzapata@usp.br

Daciberg Lima Gonçalves
dlgoncal@ime.usp.br

[1] Departamento de Matemática, IME-USP, Caixa Postal 66281-Ag. Cidade de São Paulo, São Paulo, SP CEP: 05314-970, Brazil

1 Introduction

Let $((X, \tau); Y)$ be a triple where X is a Hausdorff space, $\tau : X \to X$ is a fixed-point free involution and Y is a Hausdorff space. We say that $((X, \tau); Y)$ *satisfies the Borsuk–Ulam property* (which we shall routinely abbreviate to BUP) if for every continuous map $f : X \to Y$ there exists a point $x \in X$ such that $f(\tau(x)) = f(x)$.

Let S^m be the m-dimensional sphere, $A : S^m \to S^m$ the antipodal involution (i.e., $A(x) = -x$ for any $x \in S^m$) and $\mathbb{R}^n$ the n-dimensional Euclidean space. The famous Borsuk–Ulam theorem [3] states that for every continuous map $f : S^m \to \mathbb{R}^m$ there exists a point $x \in S^m$ such that $f(-x) = f(x)$, i.e., the triple $((S^m, A); \mathbb{R}^m)$ satisfies the Borsuk–Ulam property.

The study of BUP via sectional category is still non-existent and, in fact, this work takes a first step in this direction. Several examples are presented to illustrate the result arising in this field. We demonstrate that the triple $((X, \tau); Y)$ satisfies the BUP if $\mathrm{secat}(q) > \mathrm{secat}(q^Y)$ (Theorem 3.14). As a result, we give an alternative proof of the fact that the triple $((S^m, A); \mathbb{R}^n)$ satisfies the BUP for any $1 \leq n \leq m$ (Corollary 3.16). Moreover, we show that if $\mathrm{secat}(q : X \to X/\tau) > \mathrm{Emb}(Y)$ then the triple $((X, \tau); Y)$ satisfies the BUP, where $\mathrm{Emb}(M)$ is the smallest dimension of Euclidean spaces in which M can be embedded (Proposition 3.19). For any planar graph Γ such that $F(\Gamma, 2)$ is path-connected, we show that the triple $((S^m, A); \Gamma)$ satisfies the BUP for any $m \geq 2$ (Example 3.21). In addition, we study two natural generalizations of the Borsuk–Ulam theorem as follows.

The first natural generalization of the Borsuk–Ulam theorem consists in replacing $\mathbb{R}^n$ by a Hausdorff space Y, and then to ask which triples $((S^m, A); Y)$ satisfy the BUP. We obtain that if $\mathrm{secat}\left(q^Y : F(Y, 2) \to D(Y, 2)\right) < m + 1$ then the triple $((S^m, A); Y)$ satisfies the BUP (Example 3.15). For Y a path-connected topological manifold (without boundary), if $\dim(Y) \leq \dfrac{m}{2}$ we show that the triple $\left((S^{m+1}, A); Y\right)$ satisfies the BUP (Proposition 3.17). In particular, in Example 3.18, we show that the triple $((S^m, A); \Sigma)$ satisfies the BUP for any $m \geq 5$ and any connected surface Σ.

The second natural generalization of the Borsuk–Ulam theorem consists in replacing S^m by a connected, m-dimensional CW complex M^m and A by a free cellular involution τ defined on M^m, and then to ask which triples $((M^m, \tau); \mathbb{R}^n)$ satisfy the BUP. From [9, Lemma 2.4] or Lemma 3.28, if $n > m$ the BUP does not hold for $((M^m, \tau); \mathbb{R}^n)$. A major problem is to find the greatest $n \leq m$ such that the BUP holds for a specific (M^m, τ). Such greatest integer n is known as the *index* of τ on M^m (Definition 2.2). We present a new lower bound for the index in terms of sectional category. Indeed, we demonstrate that the index of τ on X is at least $\mathrm{secat}(q : X \to X/\tau) - 1$ (Theorem 3.30). This lower bound can be achieved. Corollary 3.31 shows that the index of τ on M^m is m when $\mathrm{secat}(q) = m + 1$. Moreover, Proposition 3.36 shows that the index of τ on M^m is $m - 1$ when $\mathrm{secat}(q : M^m \to M^m/\tau) = m$. We conjecture that the index of (M, τ) always coincides with the sectional category of the quotient map $q : M \to M/\tau$ minus 1. Several examples are presented to support this conjecture (Example 3.38).

The paper is organized as follows: In Sect. 2, we recall the notion of Borsuk–Ulam property. A key result in this paper is a topological characterisation for the BUP (it is given in Proposition 2.3) together with Remark 2.4. In Sect. 3, we begin by recalling the notions of sectional category, L–S category, category of maps and basic results about these numerical invariants. We show that the equality $\mathrm{secat}(p \times 1_Z) = \mathrm{secat}(p)$ holds for any fibration $p : E \to B$ and any topological space Z (Proposition 3.6). In [15], the author shows that $\mathrm{secat}(q^{\mathbb{R}^n}) = n$ for any $n \geq 1$. We present an alternative proof of this fact (see Lemma 3.12). In this section we study the BUP property for the triple $((X, \tau); Y)$ via sectional category. In particular, we demonstrate that the triple $((X, \tau); Y)$ satisfies the BUP if $\mathrm{secat}(q) > \mathrm{secat}(q^Y)$ (Theorem 3.14). We give several examples which extend know results about the BUP. In particular, we recover the famous Borsuk–Ulam theorem (Corollary 3.16). In addition, in Lemma 3.28, for connected topological manifolds X and Y with dimension n and $n + 1$, respectively (with $n \geq 1$); and a fixed-point free involution $\tau : X \to X$, we show that the triple $((X, \tau); Y)$ does not satisfy the BUP. As another application of our result, we present a new lower bound for the index in terms of sectional category (Theorem 3.30).

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2 Borsuk–Ulam Theory Revisited

Let $((X, \tau); Y)$ be a triple where X is a Hausdorff space, $\tau : X \to X$ is a fixed-point free involution and Y is a Hausdorff space. We say that $((X, \tau); Y)$ *satisfies the Borsuk–Ulam property* (which we shall routinely abbreviate to BUP) if for every continuous map $f : X \to Y$ there exists a point $x \in X$ such that $f(\tau(x)) = f(x)$.

Let S^m be the m-dimensional sphere, $A : S^m \to S^m$ the antipodal involution and $\mathbb{R}^n$ the n-dimensional Euclidean space. The famous Borsuk–Ulam theorem states that for every continuous map $f : S^m \to \mathbb{R}^m$ there exists a point $x \in S^m$ such that $f(x) = f(-x)$ [3].

Remark 2.1 Note that if $((X, \tau); Y)$ satisfies the BUP then the triple $((X, \tau); Z)$ also satisfies the BUP for any nonempty subspace $Z \subset Y$.

A natural generalization of the Borsuk–Ulam theorem consists in replacing S^m by a connected, m-dimensional CW complex M^m and A by a free cellular involution τ defined on M^m, and then to ask which triples $((M^m, \tau); \mathbb{R}^n)$ satisfy the BUP. From [9, Lemma 2.4], if $n > m$ the BUP does not hold for $((M^m, \tau); \mathbb{R}^n)$. A major problem is to find the greatest $n \leq m$ such that the BUP holds for a specific (M^m, τ).

Definition 2.2 Let (X, τ) be a $\mathbb{Z}_2$-space. Following [12, Definition 5.3.1] the $\mathbb{Z}_2$-index of (X, τ) is defined by:

$$\mathrm{ind}_{\mathbb{Z}_2}(X) := \min\{n \in \{0, 1, 2, \ldots\} : X \overset{\mathbb{Z}_2}{\to} S^n\}.$$

From [9, Propostion 2.2] follows that the greatest n such that $((M^m, \tau); \mathbb{R}^n)$ satisfy the BUP coincides with $\mathrm{ind}_{\mathbb{Z}_2}(M^m)$, where the $\mathbb{Z}_2$ action is given by τ.

We will fix some notation that will be used throughout the paper. The *ordered configuration space* of 2 distinct points on Y (see [6]) is the topological space

$$F(Y,2) = \{(y_1, y_2) \in Y \times Y : \ y_1 \neq y_2\}$$

topologised as a subspace of the Cartesian power $Y \times Y$. Consider the double cover $q^Y : F(Y,2) \to D(Y,2)$ from the ordered configuration space $F(Y,2)$ to its unordered quotient $D(Y,2) = F(Y,2)/\Sigma_2$ given by the obvious action τ_2 of the symmetric group Σ_2 on 2 letters. Note that the existence of a free action of $\mathbb{Z}_2$ on X is equivalent to that of a fixed-point free involution $\tau : X \to X$. In this case, with X Hausdorff, the quotient map $q : X \to X/\tau$ is a 2-sheeted covering map. Recall that a covering map is a locally trivial bundle whose fiber is a discrete space [1, Example 4.5.3, pg. 126] and thus the quotient map $q : X \to X/\tau$ is a principal fibering in the sense of Schwarz [17, Pg. 59].

It is easy to check the following topological criterion for the BUP (c.f. [8, Lemma 5]).

Proposition 2.3 *The triple $((X,\tau);Y)$ does not satisfy the BUP if and only if there exists a $\mathbb{Z}_2$-equivariant continuous map $\varphi : X \to F(Y,2)$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & F(Y,2) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q^Y} \\ X/\tau & \xrightarrow[\ \overline{\varphi}\ ]{} & D(Y,2) \end{array}$$

where $q : X \to X/\tau$ and $q^Y : F(Y,2) \to D(Y,2)$ are the 2-sheeted covering maps, and $\overline{\varphi}$ is induced by φ in the quotient spaces.

Proof Suppose first that $((X,\tau);Y)$ does not satisfy the BUP. Then there exists a map $f : X \to Y$ such that $f(x) \neq f(\tau(x))$ for all $x \in X$. Define the map $\varphi : X \to F(Y,2)$ by $\varphi(x) = (f(x), f(\tau(x)))$. Note that φ is $\mathbb{Z}_2$-equivariant, and so induces a map $\overline{\varphi}$ of the corresponding quotient spaces. Moreover, the equality $\overline{\varphi} \circ q = q^Y \circ \varphi$ holds.

We now prove the converse. Suppose that there exists a such $\mathbb{Z}_2$-equivariant map $\varphi : X \to F(Y,2)$. Let $\varphi = (\varphi_1, \varphi_2)$, and note that $\varphi(\tau(x)) = (\varphi_2(x), \varphi_1(x))$ for all $x \in X$, and so $\varphi_1(\tau(x)) = \varphi_2(x)$ for all $x \in X$. Then $\varphi_1 : X \to Y$ is a map with $\varphi_1(x) \neq \varphi_2(x) = \varphi_1(\tau(x))$ for all $x \in X$, and we have that $((X,\tau);Y)$ does not satisfy the BUP. $\square$

From [17, Pg. 61] we have the following remark.

Remark 2.4 Let X, Y be Hausdorff spaces. Suppose that X admits a fixed-point free involution τ. Note that, any commutative diagram in the form

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & F(Y,2) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q^Y} \\ X/\tau & \xrightarrow[\overline{\varphi}]{} & D(Y,2) \end{array}$$

where $\varphi : X \to F(Y, 2)$ is a continuous $\mathbb{Z}_2$-equivariant map and $\overline{\varphi}$ is induced by φ in the quotient spaces, is a pullback since φ restricts to a homeomorphism on each fiber (in this case both $q : X \to X/\tau$ and $q^Y : F(Y, 2) \to D(Y, 2)$ are 2-sheeted covering maps).

3 Sectional Category

In this section we begin by recalling the notion of sectional category together with basic results about this numerical invariant. We shall follow the terminology in [21]. If f is homotopic to g we shall denote by $f \simeq g$. The map $1_Z : Z \to Z$ denotes the identity map. Fibrations are taken in the Hurewicz sense.

Let $p : E \to B$ be a fibration. A *cross-section* or *section* of p is a right inverse of p, i.e., a map $s : B \to E$, such that $p \circ s = 1_B$. Moreover, given a subspace $A \subset B$, a *local section* of p over A is a section of the restriction map $p_| : p^{-1}(A) \to A$, i.e., a map $s : A \to E$, such that $p \circ s$ is the inclusion $A \hookrightarrow B$.

We recall the following definition, see [16] or [17].

Definition 3.1 The *sectional category* of p, called originally by Schwarz genus of p, and denoted by $\text{secat}(p)$, is the minimal cardinality of open covers of B, such that each element of the cover admits a continuous local section to p. We set $\text{secat}(p) = \infty$ if no such finite cover exists.

For a commutative ring R and a proper ideal $S \subset R$, the *nilpotency index* of S is given by

$$\text{nil}(S) = \min\{k : \ \text{any product of } k \text{ elements of } S \text{ is trivial}\}.$$

Note that, $\text{nil}(S)$ coincides with $n + 1$, where n is the maximum number of factors in a nonzero product of elements from S.

The following statement gives a lower bound in terms of any multiplicative cohomology (see [20, Proposição 4.3.17-(3), pg. 138]).

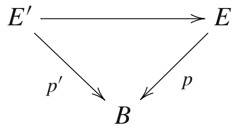
Lemma 3.2 *Let h^* be a multiplicative cohomology theory and $p : E \to B$ be a fibration, then*

$$\text{secat}(p) \geq \text{nil}(\text{Ker}(p^*)),$$

where $p^* : h^*(B) \to h^*(E)$ is the induced homomorphism in cohomology.

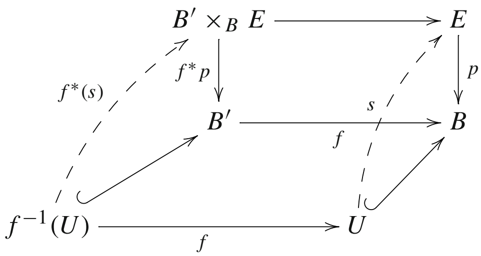
Remark 3.3 Lemma 3.2 implies that if there exist cohomology classes $\alpha_1, \ldots, \alpha_k \in h^*(B)$ with $p^*(\alpha_1) = \cdots = p^*(\alpha_k) = 0$ and $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$, then $\mathrm{secat}(p) \geq k+1$. In this paper we will use Lemma 3.2 for h^* as being the singular cohomology with any coefficient ring (as was presented by James in [11, pg. 342]).

Now, note that, if the following diagram



commutes up homotopy, then $\mathrm{secat}(p') \geq \mathrm{secat}(p)$ (see [17, Proposition 6, pg. 70]). Also, from [17, Proposition 7, pg. 71], for any fibration $p : E \to B$ and any continuous map $f : B' \to B$, note that any local section $s : U \to E$ of $p : E \to B$ induces a local section of the canonical pullback $f^*p : B' \times_B E \to B'$, called *the local pullback section* $f^*(s) : f^{-1}(U) \to B' \times_B E$, simply by defining

$$f^*(s)(b') = \big(b', (s \circ f)\,(b')\big)\,.$$

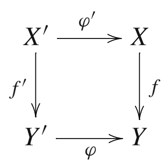


Thus,

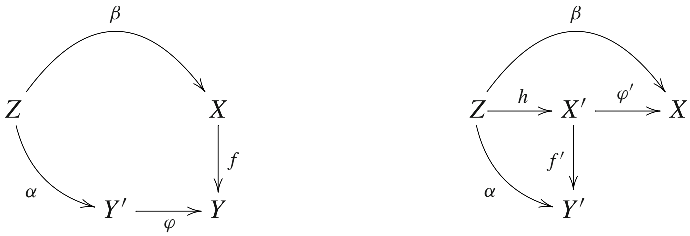
$$\mathrm{secat}(f^*p) \leq \mathrm{secat}(p). \tag{3.1}$$

Before stating Lemma 3.5, let us present a key concept which we call *quasi pullback*.

Definition 3.4 By a *quasi pullback* we mean a strictly commutative diagram



such that, for any strictly commutative diagram as the one on the left hand-side of (3.2), there exists a (not necessarily unique) continuous map $h : Z \to X'$ that renders a strictly commutative diagram as the one on the right hand-side of (3.2).

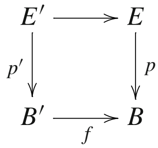


(3.2)

Note that such a condition amounts to saying that X' contains the canonical pullback $Y' \times_Y X$ determined by f and φ as a retract in a way that is compatible with the mappings into X and Y'.

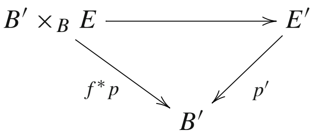
Then we have the following statement.

Lemma 3.5 *Let $p : E \to B$ be a fibration. If the following square*

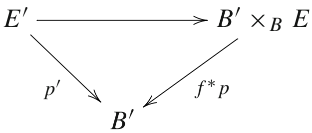


is a quasi pullback, then $\mathrm{secat}(p') \leq \mathrm{secat}(p)$.

Proof Since p' is a quasi pullback, we have the following commutative triangle



and thus $\mathrm{secat}(f^*p) \geq \mathrm{secat}(p')$. Similarly, since f^*p is the canonical pullback, we have the following commutative triangle



and thus $\mathrm{secat}(p') \geq \mathrm{secat}(f^*p)$. Hence, the equality $\mathrm{secat}(p') = \mathrm{secat}(f^*p)$ holds. By the inequality (3.1), we obtain $\mathrm{secat}(p') \leq \mathrm{secat}(p)$. □

In Example 3.38 we will use the equality $\mathrm{secat}(p \times 1_Z) = \mathrm{secat}(p)$ which holds for any fibration $p : E \to B$ and any topological space Z. For that reason, we present a proof to it.

Proposition 3.6 *Let $p : E \to B$ be a fibration and Z be a topological space, then*

$$\mathrm{secat}(p \times 1_Z) = \mathrm{secat}(p).$$

Proof Note that, if $s : U \to E$ is a section to p, then the product $s \times 1_Z : U \times Z \to E \times Z$ is a section to $p \times 1_Z$, and thus, $\mathrm{secat}(p \times 1_Z) \leq \mathrm{secat}(p)$. The other inequality follows from the fact that the square

$$\begin{array}{ccc} E & \xrightarrow{j_{z_0}} & E \times Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p\times 1_Z} \\ B & \xrightarrow[j_{z_0}]{} & B \times Z \end{array}$$

where $j_{z_0}(-) = (-, z_0)$ is the natural inclusion ($z_0 \in Z$), is a quasi pullback together with Lemma 3.5. □

Next, we recall the notion of LS category which, in our setting, is one bigger than the one given in [5, Definition 1.1, pg.1].

Definition 3.7 The *Lusternik–Schnirelmann category* (L–S category) or category of a topological space X, denoted by $\mathrm{cat}(X)$, is the least integer m such that X can be covered by m open sets, all of which are contractible within X. We set $\mathrm{cat}(X) = \infty$ if no such m exists.

We have $\mathrm{cat}(X) = 1$ iff X is contractible. The L–S category is a homotopy invariant, i.e., if X is homotopy equivalent to Y (which we shall denote by $X \simeq Y$), then $\mathrm{cat}(X) = \mathrm{cat}(Y)$. Furthermore, the invariant satisfies the following properties.

Lemma 3.8 (1) [[11], Proposition 5.1, pg. 336] *If X is a $(q-1)$-connected CW complex ($q \geq 1$), then*

$$\mathrm{cat}(X) \leq \frac{\mathrm{hdim}(X)}{q} + 1,$$

where $\mathrm{hdim}(X)$ *denotes the homotopical dimension of X, i.e., the minimal dimension of CW complexes having the homotopy type of X.*

(2) [[20], Proposição 4.1.34, pg. 108] *We have*

$$\mathrm{cat}(X) \geq \mathrm{nil}\left(\widetilde{h}^*(X)\right),$$

where $\widetilde{h}^*(X)$ is any multiplicative reduced cohomology theory.

We recall the following statements.

Lemma 3.9 *Let $p : E \to B$ be a fibration.*

(1) [17, Theorem 18, pg. 108] *We have* $\mathrm{secat}(p) \leq \mathrm{cat}(B)$.
(2) [20, Proposição 4.3.17, pg. 138] *If p is nulhomotopic, then* $\mathrm{secat}(p) = \mathrm{cat}(B)$.

Remark 3.10 Given a Hausdorff space X admitting a fixed-point free involution τ, we have that:

(i) The quotient map $q : X \to X/\tau$ is the trivial bundle if and only if it admits a continuous cross-section [18, Pg. 36]. In particular, in the case when X is Hausdorff and path-connected, the quotient map $q : X \to X/\tau$ is not the trivial bundle and does not admit a continuous cross-section.
(ii) The quotient map $q\colon X \to X/\tau$ is closed (see [4, Theorem 3.1, pg. 38]). From [13, Corollary 1, pg. 823] if X is paracompact, then X/τ is also paracompact and thus $q\colon X \to X/\tau$ is a fibration.

Example 3.11 Recall that the $\mathbb{Z}_2$-cohomology of $\mathbb{R}P^m$ ($m \geq 1$) is given by $H^*(\mathbb{R}P^m; \mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[\alpha]}{\langle \alpha^{m+1} \rangle}$ with $0 \neq \alpha \in H^1(\mathbb{R}P^m; \mathbb{Z}_2)$. Then, for dimensional reasons, the induced homomorphism $q^*_{\mathbb{Z}_2} : H^*(\mathbb{R}P^m; \mathbb{Z}_2) \to H^*(S^m; \mathbb{Z}_2)$ is trivial and thus $\mathrm{Ker}(q^*_{\mathbb{Z}_2}) = \widetilde{H}^*(\mathbb{R}P^m; \mathbb{Z}_2)$ for any $m \geq 2$. Then $\mathrm{Nil}\left(\mathrm{Ker}(q^*_{\mathbb{Z}_2})\right) \geq m+1$ for any $m \geq 2$. In addition $\mathrm{cat}(\mathbb{R}P^m) = m+1$ (see [5, Example 1.8, pg.4]). Thus, for any $m \geq 2$, we have

$$\mathrm{secat}(q : S^m \to \mathbb{R}P^m) = \mathrm{cat}(\mathbb{R}P^m) = \mathrm{Nil}\left(\mathrm{Ker}(q^*_{\mathbb{Z}_2})\right) = m+1,$$

where $q^*_{\mathbb{Z}_2} : H^*(\mathbb{R}P^m; \mathbb{Z}_2) \to H^*(S^m; \mathbb{Z}_2)$ is the induced homomorphism in $\mathbb{Z}_2$-cohomology. Furthermore, $\mathrm{secat}(q : S^1 \to \mathbb{R}P^1) = \mathrm{cat}(\mathbb{R}P^1) = 2$. Indeed, for $m \geq 2$, by Lemma 3.2 together with Lemma 3.9 we have

$$\begin{aligned} m+1 &\leq \mathrm{Nil}\left(\mathrm{Ker}(q^*_{\mathbb{Z}_2})\right) \\ &\leq \mathrm{secat}(q : S^m \to \mathbb{R}P^m) \\ &\leq \mathrm{cat}(\mathbb{R}P^m) \\ &= m+1. \end{aligned}$$

On the other hand, $\mathrm{secat}(q : S^1 \to \mathbb{R}P^1) \geq 2$. Moreover, again by Lemma 3.9, $\mathrm{secat}(q : S^1 \to \mathbb{R}P^1) \leq \mathrm{cat}(\mathbb{R}P^1) = 2$.

In [15], the author shows that $\mathrm{secat}(q^{\mathbb{R}^n}) = n$ for any $n \geq 1$. However, we present an alternative proof of this fact.

Lemma 3.12 *We have that* $\mathrm{secat}\left(q^{\mathbb{R}^n} : F(\mathbb{R}^n, 2) \to D(\mathbb{R}^n, 2)\right) = n$ *for any* $n \geq 1$.

Proof The case $n = 1$ is easy since the configuration space $D(\mathbb{R}, 2)$ is contractible. For $n \geq 2$, consider the maps $\varphi\colon S^{n-1} \to F(\mathbb{R}^n, 2)$ and $\psi\colon F(\mathbb{R}^n, 2) \to S^{n-1}$ given by $\varphi(x) = (x, -x)$ and $\psi(x, y) = \dfrac{x-y}{\| x-y \|}$. Then, we have that the following diagrams

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi} & F(\mathbb{R}^n, 2) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q^{\mathbb{R}^n}} \\ \mathbb{R}P^{n-1} & \xrightarrow[\overline{\varphi}]{} & D(\mathbb{R}^n, 2) \end{array} \qquad \begin{array}{ccc} F(\mathbb{R}^n, 2) & \xrightarrow{\psi} & S^{n-1} \\ {\scriptstyle q^{\mathbb{R}^n}}\downarrow & & \downarrow{\scriptstyle q} \\ D(\mathbb{R}^n, 2) & \xrightarrow[\overline{\psi}]{} & \mathbb{R}P^{n-1} \end{array}$$

are pullbacks. Then, by Lemma 3.5, we conclude that $\mathrm{secat}(q^{\mathbb{R}^n}) = \mathrm{secat}(q)$ and therefore, $\mathrm{secat}(q^{\mathbb{R}^n}) = n$ (by Example 3.11). ◻

In the same way, we will check that $\mathrm{secat}\left(q^{S^n}\right) = n+1$ for any $n \geq 1$.

Lemma 3.13 *We have that* $\mathrm{secat}\left(q^{S^n} : F(S^n, 2) \to D(S^n, 2)\right) = n+1$ *for any* $n \geq 1$.

Proof Consider the following pullbacks

$$\begin{array}{ccc} S^{n} & \xrightarrow{\varphi} & F(S^n, 2) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q^{S^n}} \\ \mathbb{R}P^{n} & \xrightarrow[\overline{\varphi}]{} & B(S^n, 2) \end{array} \qquad \begin{array}{ccc} F(S^n, 2) & \xrightarrow{\psi} & S^{n} \\ {\scriptstyle q^{S^n}}\downarrow & & \downarrow{\scriptstyle q} \\ B(S^n, 2) & \xrightarrow[\overline{\psi}]{} & \mathbb{R}P^{n} \end{array}$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = \dfrac{x-y}{\| x-y \|}$. Then, by Lemma 3.5, we conclude that $\mathrm{secat}(q^{S^n}) = \mathrm{secat}(q)$ and therefore, $\mathrm{secat}(q^{S^n}) = n+1$ (by Example 3.11). ◻

Our main result is as follows. We show that the BUP is valid if certain conditions in terms of sectional category are satisfied.

Theorem 3.14 (Principal theorem) *Suppose that X and Y are Hausdorff spaces with X and $F(Y, 2)$ paracompact. If the triple $((X, \tau); Y)$ does not satisfy the BUP then*

$$\mathrm{secat}(q) \leq \mathrm{secat}(q^Y).$$

Equivalently, if $\mathrm{secat}(q) > \mathrm{secat}(q^Y)$ *then the triple $((X, \tau); Y)$ satisfies the BUP.*

Proof It follows from Proposition 2.3 and Remark 2.4 together with Lemma 3.5. ◻

Next, we will present direct applications of Theorem 3.14.

Example 3.15 Let $m \geq 1$ and Y be a Hausdorff space with $F(Y, 2)$ paracompact, and consider the covering maps:

$$\begin{array}{c} S^m \\ {\scriptstyle q}\downarrow \\ \mathbb{R}P^m \end{array} \qquad \begin{array}{c} F(Y, 2) \\ \downarrow{\scriptstyle q^Y} \\ D(Y, 2) \end{array}$$

Recall that $\operatorname{secat}(q) = m + 1$ (see Example 3.11). Therefore, by Theorem 3.14, if $\operatorname{secat}(q^Y) < m + 1$ then the triple $((S^m, A); Y)$ satisfies the BUP.

Example 3.15 implies the famous Borsuk–Ulam theorem [3].

Corollary 3.16 *(Famous Borsuk–Ulam theorem) We have that the triple* $((S^m, A); \mathbb{R}^n)$ *satisfies the BUP for any* $1 \leq n \leq m$.

Proof We have that $\operatorname{secat}(q^{\mathbb{R}^n}) = n$ (see Lemma 3.12), and thus, from Example 3.15, we conclude that the triple $((S^m, A); \mathbb{R}^n)$ satisfies the BUP for any $1 \leq n \leq m$. □

Moreover, we have the following statement.

Proposition 3.17 *Let Y be a path-connected topological manifold (without boundary) such that* $\dim(Y) \leq \frac{m}{2}$. *Then*

(1) *The triple* $\big((S^{m+1}, A); Y\big)$ *satisfies the BUP.*
(2) *If* $\operatorname{hdim}(D(Y, 2)) \leq 2\dim(Y) - 1$ *then,* $((S^m, A); Y)$ *satisfies the BUP.*

Proof The case, $\dim(Y) \leq 1$ is shown easily. We will suppose that $\dim(Y) \geq 2$.

(1) From Lemma 3.9 together with Lemma 3.8-item (1), we have

$$\begin{aligned} \operatorname{secat}(q^Y) &\leq \operatorname{cat}(D(Y, 2)) \\ &\leq \operatorname{hdim}(D(Y, 2)) + 1 \\ &\leq 2\dim(Y) + 1 \\ &\leq m + 1. \end{aligned}$$

Thus $\operatorname{secat}(q^Y) < m + 2 = \operatorname{secat}(q\colon S^{m+1} \to \mathbb{R}P^{m+1})$ and by Example 3.15, then we conclude that the triple $\big((S^{m+1}, A); Y\big)$ satisfies the BUP.

(2) From Lemma 3.9 together with Lemma 3.8-item (1), we have

$$\begin{aligned} \operatorname{secat}(q^Y) &\leq \operatorname{cat}(D(Y, 2)) \\ &\leq \operatorname{hdim}(D(Y, 2)) + 1 \\ &\leq 2\dim(Y) - 1 + 1 \\ &\leq m. \end{aligned}$$

Thus $\operatorname{secat}(q^Y) < m + 1 = \operatorname{secat}(q\colon S^m \to \mathbb{R}P^m)$ and by Example 3.15, then we conclude that the triple $((S^m, A); Y)$ satisfies the BUP.

□

We do not know an example of a path-connected manifold Y (without boundary) such that $\operatorname{hdim}(D(Y, 2)) = 2\dim(Y)$.

An immediate consequence of Proposition 3.17 is the following example.

Example 3.18 Let Σ be a connected surface. Then the triple $((S^m, A); \Sigma)$ satisfies the BUP for any $m \geq 5$.

Example 3.18 together with [7, Theorem 2, pg. 1743] imply that the triple $\big((S^m, A); \mathbb{R}P^2\big)$ satisfies the BUP for any $m \geq 2$. In [7, Theorem 2, pg. 1743], the authors showed that the triple $\big((S^m, A); \mathbb{R}P^2\big)$ satisfies the BUP for $m \in \{2, 3\}$. On the other hand, we can check that (or see [7, Proposition 4, pg. 1743]) the triple $((S^m, A); \Sigma)$ satisfies the BUP for any $m \geq 2$ and any connected closed surface Σ different from $\mathbb{R}P^2$ and S^2.

Let $\text{Emb}(M)$ be the smallest dimension of Euclidean spaces when M can be embedded. We set $\text{Emb}(M) = \infty$ if no such embedding exists. We have the following statement.

Proposition 3.19 *Suppose that X and Y are Hausdorff spaces with* $\text{Emb}(Y) < \infty$. *If* $\text{secat}(q : X \to X/\tau) > \text{Emb}(Y)$ *then the triple* $((X, \tau); Y)$ *satisfies the BUP.*

Proof Consider $Y \subset \mathbb{R}^k$, where $k = \text{Emb}(Y)$. We have the following pullback

$$\begin{array}{ccc} F(Y,2) & \longrightarrow & F(\mathbb{R}^k,2) \\ {\scriptstyle q^Y}\big\downarrow & & \big\downarrow{\scriptstyle q^{\mathbb{R}^k}} \\ D(Y,2) & \longrightarrow & D(\mathbb{R}^k,2) \end{array}$$

where the horizontal maps in the diagram are induced by the inclusion $Y \hookrightarrow \mathbb{R}^k$. Then $\text{secat}(q^Y) \leq \text{secat}(q^{\mathbb{R}^k}) = k = \text{Emb}(Y) < \text{secat}(q : X \to X/\tau)$. Thus, by Theorem 3.14, we obtain that the triple $((X, \tau); Y)$ satisfies the BUP. □

In particular, we have the following example.

Example 3.20 Note that $\text{Emb}(S^2) = 3$. Thus, by Proposition 3.19, we obtain that the triple $\big((S^m, A); S^2\big)$ satisfies the BUP for any $m \geq 3$.

Note that, Example 3.20 cannot be improved, that is, we can check that (or see [7, Corollary 1-(b), pg. 1743]) the triple $\big((S^2, A); S^2\big)$ does not satisfy the BUP.

The following statement is a partial result of the BUP when $Y = \Gamma$ is a graph.

Example 3.21 Suppose that Γ is a planar graph such that $F(\Gamma, 2)$ is path-connected. We have that $\text{Emb}(\Gamma) = 2$. Thus, by Proposition 3.19, we obtain that the triple $((S^m, A); \Gamma)$ satisfies the BUP for any $m \geq 2$. Moreover, note that $\text{secat}(q^\Gamma) = 2$. Indeed, the inequality $2 \leq \text{secat}(q^\Gamma)$ follows from Remark 3.10 item (i) together with the hypothesis that $F(\Gamma, 2)$ is path-connected. The inequality $\text{secat}(q^\Gamma) \leq 2$ follows from the hypothesis that Γ is planar (hence $\text{secat}(q^\Gamma) \leq \text{secat}(q^{\mathbb{R}^2}) = 2$).

Remark 3.22 Note that, for $m \geq 1$, by the famous Borsuk–Ulam theorem (Example 3.16) together with Remark 2.1, we obtain that the triple $((S^m, A); Z)$ satisfies the BUP for any subspace $Z \subset \mathbb{R}^m$.

When S^∞ is the infinite dimensional sphere we have:

Example 3.23 For cohomological reasons, note that $\mathrm{secat}(S^\infty \to \mathbb{R}P^\infty) = \infty$, then the triple $((S^\infty, A); Y)$ satisfies the BUP for any finite dimensional topological manifold Y. Indeed, we have

$$\begin{aligned}\mathrm{secat}(q^Y) &\le \mathrm{cat}(D(Y,2))\\ &\le \dim\,(D(Y,2)) + 1\\ &< \infty.\end{aligned}$$

Next, we will study the BUP when $X = F(Z, 2)$ is the ordered configuration space. Recall that, we have the free involution $\tau_2 : F(Z, 2) \to F(Z, 2),\ \tau_2(x, y) = (y, x)$ and the equality $\mathrm{secat}(q^{\mathbb{R}^n}) = n$ (see Lemma 3.12).

Proposition 3.24 *Let Y be a path-connected topological manifold (without boundary). If $\dim(Y) \le \dfrac{n-1}{2}$ then the triple $\big((F(\mathbb{R}^{n+1}, 2), \tau_2); Y\big)$ satisfies the BUP.*

Proof Let $h : F(\mathbb{R}^{n+1}, 2) \to Y$ be any continuous map. Consider $j : S^n \to F(\mathbb{R}^{n+1}, 2)$ given by $j(x) = (x, -x)$ and we have the composition $h \circ j : S^n \to Y$. Then, by Proposition 3.17 item (1), there exists $x \in S^n$ such that $h \circ j(-x) = h \circ j(x)$, and thus $h(-x, x) = h(x, -x)$. Then, there is a point $z = (x, -x) \in F(\mathbb{R}^{n+1}, 2)$ such that $h(\tau_2(z)) = h(z)$. Therefore, the triple $\big((F(\mathbb{R}^{n+1}, 2), \tau_2); Y\big)$ satisfies the BUP. □

In particular, when Y is a surface, we have the following statement.

Example 3.25 Let Σ be a path-connected surface, then the triple $((F(\mathbb{R}^n, 2), \tau_2); \Sigma)$ satisfies the BUP for any $n \ge 6$ (by Proposition 3.24). Furthermore, by Proposition 3.19, we obtain that $((F(\mathbb{R}^n, 2), \tau_2); \Sigma)$ satisfies the BUP for any $n \ge 4$ when Σ is a connected orientable surface.

Lemma 3.13 says that the equality $\mathrm{secat}(q^{S^n}) = n + 1$ holds for any $n \ge 1$. Thus, we have the BUP when $X = F(S^n, 2)$.

Proposition 3.26 *Let Y be a path-connected topological manifold (without boundary). If $\dim(Y) \le \dfrac{n}{2}$ then the triple $\big((F(S^{n+1}, 2), \tau_2); Y\big)$ satisfies the BUP.*

Proof Let $h : F(S^{n+1}, 2) \to Y$ be any continuous map. Consider $j : S^{n+1} \to F(S^{n+1}, 2)$ given by $j(x) = (x, -x)$ and we have the composition $h \circ j : S^{n+1} \to Y$. Then, by Proposition 3.17, there exists $x \in S^{n+1}$ such that $h \circ j(-x) = h \circ j(x)$, and thus $h(-x, x) = h(x, -x)$. Then, there is a point $z = (x, -x) \in F(S^{n+1}, 2)$ such that $h(\tau_2(z)) = h(z)$. Therefore, the triple $\big((F(S^{n+1}, 2), \tau_2); Y\big)$ satisfies the BUP. □

One more time, when Y is a surface, we have the following statement.

Example 3.27 Let Σ be a path-connected surface. By Proposition 3.26, then the triple $((F(S^n, 2), \tau_2); \Sigma)$ satisfies the BUP for any $n \ge 5$. On the other hand, by Proposition 3.19, we obtain $((F(S^n, 2), \tau_2); \Sigma)$ satisfies the BUP for any $n \ge 3$ when Σ is a connected orientable surface.

The following statement generalises [9, Lemma 2.4]. Note that for any principal $\mathbb{Z}_2$-bundle $q : M^m \to M^m/\tau$, we will write $f_q : M^m/\tau \to \mathbb{R}P^\infty = B\mathbb{Z}_2$ for the classifying map of the bundle q. It is unique up to homotopy.

Lemma 3.28 *Let X be a path-connected CW complex and Y be a path-connected topological manifold with dimension n and $n+1$, respectively (with $n \geq 1$); and $\tau : X \to X$ be a fixed-point free cellular involution. Then $((X,\tau);Y)$ does not satisfy the BUP.*

Proof Since $\dim(X) = n$ and thus $\dim(X/\tau) = n$, there is a map $\overline{\rho}\colon X/\tau \to \mathbb{R}P^n$ such that the triangle

$$\begin{array}{ccc} X & & S^n \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ X/\tau & \xrightarrow{\overline{\rho}} & \mathbb{R}P^n \\ & \searrow^{f_q} & \downarrow \\ & & \mathbb{R}P^\infty \end{array}$$

commutes up homotopy. Then, there is a $\mathbb{Z}_2$-equivariant continuous map $\rho\colon X \to S^n$ such that $q' \circ \rho = \overline{\rho} \circ q$. Let $\mathbb{R}^{n+1} \subset Y$ be an embedding. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{\rho} & S^n & \xrightarrow{\phi} & F(\mathbb{R}^{n+1},2) & \xhookrightarrow{i} & F(Y,2) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q^{\mathbb{R}^{n+1}}} & & \downarrow{\scriptstyle q^Y} \\ X/\tau & \xrightarrow{\overline{\rho}} & \mathbb{R}P^n & \xrightarrow{\overline{\phi}} & D(\mathbb{R}^{n+1},2) & \xhookrightarrow{\overline{i}} & D(Y,2) \end{array}$$

where $\phi(x) = (x, -x)$ and i is induced by the inclusion $\mathbb{R}^{n+1} \hookrightarrow Y$. Then $\varphi : X \to F(Y,2)$ given by $\varphi = i \circ \phi \circ \rho$ is a $\mathbb{Z}_2$-equivariant continuous map such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & F(Y,2) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q^Y} \\ X/\tau & \xrightarrow[\overline{\varphi}]{} & D(Y,2) \end{array}$$

and we conclude that $((X,\tau);Y)$ does not satisfy the BUP. □

The following statement presents estimates of the sectional category $\text{secat}(q^Y)$.

Proposition 3.29 *If Y is a connected topological manifold (without boundary) with dimension n ($n \geq 1$), then*

(1) $n \leq \operatorname{secat}(q^Y) \leq 2n+1$.
(2) *If* $\operatorname{hdim}(D(Y,2)) \leq 2\dim(Y)-1$ *then,* $n \leq \operatorname{secat}(q^Y) \leq 2n$.

Proof The case, $n = 1$ is shown easily. We will suppose that $n \geq 2$. Let $\mathbb{R}^n \subset Y$ be an embedding. We have the following pullback:

$$\begin{array}{ccc} F(\mathbb{R}^n, 2) & \hookrightarrow & F(Y, 2) \\ {\scriptstyle q^{\mathbb{R}^n}}\downarrow & & \downarrow{\scriptstyle q^Y} \\ D(\mathbb{R}^n, 2) & \hookrightarrow & D(Y, 2) \end{array}$$

Then

$$\begin{aligned} \operatorname{secat}(q^Y) &\geq \operatorname{secat}(q^{\mathbb{R}^n}) \\ &= n. \end{aligned}$$

On the other hand,

(1) by Lemma 3.8, we have

$$\begin{aligned} \operatorname{secat}(q^Y) &\leq \operatorname{cat}(D(Y,2)) \\ &\leq \operatorname{hdim}(D(Y,2))+1 \\ &\leq 2n+1. \end{aligned}$$

(2) By Lemma 3.8 together with the hypotheses $\operatorname{hdim}(D(Y,2)) \leq 2\dim(Y)-1$, we have

$$\begin{aligned} \operatorname{secat}(q^Y) &\leq \operatorname{cat}(D(Y,2)) \\ &\leq \operatorname{hdim}(D(Y,2))+1 \\ &\leq 2n-1+1 \\ &\leq 2n. \end{aligned}$$

□

Note that, any double cover with path-connected total space has sectional category at least 2. Furthermore, for any connected m-dimensional CW complex M^m and τ be a free cellular involution defined on M^m, the inequalities $2 \leq \operatorname{secat}(q : M^m \to M^m/\tau) \leq m+1$ hold.

Now, we present a new lower bound for the index of (M^m, τ) in terms of the sectional category $\operatorname{secat}(q : M^m \to M^m/\tau)$.

Theorem 3.30 *Let* X *be a Hausdorff paracompact space admitting a fixed-point free involution* τ. *Let* $q : X \to X/\tau$ *be the quotient map. If* $1 \leq n \leq \operatorname{secat}(q)-1$, *then* *the triple* $((X,\tau);\mathbb{R}^n)$ *satisfies the BUP. In particular, the index of* τ *on* X *is at least* $\mathrm{secat}(q)-1$.

Proof By Theorem 3.14 we have that $\big((X,\tau);\mathbb{R}^{\mathrm{secat}(q)-1}\big)$ satisfies the BUP. □

This lower bound can be achieved. Example 3.16 shows that the index of the antipodal involution A on S^m is $m=\mathrm{secat}(q)-1$. More general, we have the following statement.

Corollary 3.31 *If* $\mathrm{secat}(q: M^m \to M^m/\tau) = m+1$ *then the index of* τ *on* M^m *is* m.

Proof It follows from Theorem 3.30 together with the fact that the index is at most m (see [9, Lemma 2.4] or Lemma 3.28). □

The *category* of a map $f : X \to Y$, denote $\mathrm{cat}(f)$, is the least integer m such that X can be covered by m open sets $U_1,\ldots,U_m$, such that each restriction $f\,|_{U_i}$ is nullhomotopic. Note that, $\mathrm{cat}(1_X)=\mathrm{cat}(X)$. This numerical invariant was introduced by Berstein and Ganea in [2].

We recall basic properties concerning category of a map, see [2].

Proposition 3.32 (1) *If* $f \simeq g$ *then* $\mathrm{cat}(f)=\mathrm{cat}(g)$.

(2) *For any map* $f : X \to Y$*, we have* $\mathrm{cat}(f) \le \min\{\mathrm{cat}(X),\mathrm{cat}(Y)\}$.

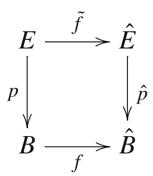
(3) *We have*

$$\mathrm{cat}(f) \ge \mathrm{Nil}\left(im\ (f^*)\right),$$

where $f^* : \widetilde{h}^*(Y) \to \widetilde{h}^*(X)$ *denotes the induced homomorphism in any multiplicative reduced cohomology theory.*

We recall from [5, Proposition 9.18, pg. 261] how sectional category relates to the category of classifying maps.

Proposition 3.33 *Suppose* $p : E \to B$ *is a fibration arising as a pullback of a fibration* $\hat{p} : \hat{E} \to \hat{B}$



where $\hat{E}$ *is contractible. Then* $\mathrm{secat}(p)=\mathrm{cat}(f)$.

Let M^m be a connected m-dimensional CW complex and A be a free cellular involution τ defined on M^m. We recall that for any principal $\mathbb{Z}_2$-bundle $q : M^m \to M^m/\tau$, we write $f_q : M^m/\tau \to \mathbb{R}P^\infty = B\mathbb{Z}_2$ for the classifying map of the bundle q. It is unique up to homotopy. Denoting the generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)\cong\mathbb{Z}_2$ by α, the characteristic class of the principal bundle is then $\gamma = f_q^*(\alpha) \in H^1(M^m/\tau;\mathbb{Z}_2)$. Since the bundle is non-trivial, it follows that $\gamma \neq 0$. In addition, by Proposition 3.33, the sectional category $\mathrm{secat}(q : M^m \to M^m/\tau) = \mathrm{cat}(f_q)$. So, we obtain the following statement.

Proposition 3.34 *Let γ be the characteristic class of the principal bundle $q : M^m \to M^m/\tau$. For $n \leq m$, if $\gamma^n \neq 0$ then the triple $((M^m, \tau); \mathbb{R}^n)$ satisfies the BUP.*

Proof From Proposition 3.33, $\mathrm{secat}(q) = \mathrm{cat}(f_q)$ where f_q is the classifying map of q. By Proposition 3.32-item (3), $\mathrm{cat}(f_q) \geq n+1$ since $\gamma \in \mathrm{im}\,(f_q^*)$ and $\gamma^n \neq 0$. Then $\mathrm{secat}(q) \geq n+1$ and thus by Theorem 3.30, we conclude that the triple $((M^m, \tau); \mathbb{R}^n)$ satisfies the BUP. □

The following implication of Proposition 3.34 was proved in [9, Theorem 3.4] for $n = m$.

Lemma 3.35 [9, Theorem 3.4] *Let γ be the characteristic class of the principal bundle $q : M^m \to M^m/\tau$. The triple $((M^m, \tau); \mathbb{R}^m)$ satisfies the BUP if and only if $\gamma^m \neq 0$.*

The following statement shows that the lower bound of the index given in Proposition 3.30 is also achieved.

Proposition 3.36 *If $\mathrm{secat}(q : M^m \to M^m/\tau) = m$, then the index of τ on M^m is $m - 1$.*

Proof From Theorem 3.14, we have the triple $((M^m, \tau); \mathbb{R}^n)$ satisfies the BUP for any $n < m$. Note that, $\gamma^m = 0$ since $\mathrm{secat}(q : M^m \to M^m/\tau) = m$. Then, by Lemma 3.35, we have that the triple $((M^m, \tau); \mathbb{R}^m)$ does not satisfy the BUP, then the index of τ on M^m is $m - 1$. □

From [5, Proposition 1.27-(2), pg. 14] we recall that $\mathrm{cat}(X \vee Y) = \max\{\mathrm{cat}(X), \mathrm{cat}(Y)\}$. Then, we have the following example which satisfies the condition of Proposition 3.36.

Example 3.37 Let $M = S^m \vee S^{m-1} \vee S^m$ with $m \geq 3$ and τ be a free cellular involution on M such that $M/\tau = \mathbb{R}P^{m-1} \vee S^m$. Indeed, τ interchanges the two S^m summands from the wedge sums and acts antipodally on the S^{m-1} summand. Similarly, like the calculation of the sectional category $\mathrm{secat}(S^{m-1} \to \mathbb{R}P^{m-1}) = m$, we have that $\mathrm{secat}(q : S^m \vee S^{m-1} \vee S^m \to \mathbb{R}P^{m-1} \vee S^m) = \mathrm{cat}(\mathbb{R}P^{m-1} \vee S^m) = \mathrm{nil}\left(\mathrm{Ker}(q^*_{\mathbb{Z}_2})\right) = m$.

From Proposition 3.6, the equality $\mathrm{secat}(p \times 1_Z) = \mathrm{secat}(p)$ holds for any fibration. Thus, we have the following example.

Example 3.38 For any positive integers m and k such that $2 \leq k \leq m+1$ consider the m-dimensional smooth manifold $M^m = S^{k-1} \times S^1 \times \cdots \times S^1$ (product of one S^{k-1} and $m-k+1$ copies of S^1) equipped with the free involution $\tau = A \times 1_{S^1} \times \cdots \times 1_{S^1}$ (A the antipodal involution on S^{k-1}). Note that, the quotient map $q' : M^m \to M^m/\tau$ coincidences with the product $q \times 1_{S^1} \times \cdots \times 1_{S^1}$, where $q : S^{k-1} \to \mathbb{R}P^{k-1}$ is the usual 2-covering map, and so, by Proposition 3.6, we obtain that $\mathrm{secat}(q') = \mathrm{secat}(q) = k$. On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} S^{k-1} \times S^1 \times \cdots \times S^1 & \xrightarrow{\quad\varphi\quad} & F(\mathbb{R}^k, 2) \\ {\scriptstyle q \times 1_{S^1} \times \cdots \times 1_{S^1}} \downarrow & & \downarrow {\scriptstyle q^{\mathbb{R}^k}} \\ \mathbb{R}P^{k-1} \times S^1 \times \cdots \times S^1 & \xrightarrow[\quad\overline{\varphi}\quad]{} & D(\mathbb{R}^k, 2) \end{array}$$

where $\varphi(x, z_1, \ldots, z_{n-k+1}) = (x, -x)$ for any $(x, z_1, \ldots, z_{n-k+1}) \in S^{k-1} \times S^1 \times \cdots \times S^1$, and thus the triple $\left(\left(S^{k-1} \times S^1 \times \cdots \times S^1, A \times 1_{S^1} \times \cdots \times 1_{S^1}\right); \mathbb{R}^k\right)$ does not satisfy the BUP. Therefore, by Proposition 3.30, the index of (M^m, τ) is equal to $k - 1 = \mathrm{secat}(q') - 1$ (compare with [10, pg. 772]).

Motivated by Theorem 3.30, Corollary 3.31, Proposition 3.36, and Example 3.38 we formulate the following conjecture.

Conjecture. Let M^m be a connected, m-dimensional CW complex and τ be a free cellular involution defined on M^m. The index of τ on M^m is equal to $\mathrm{secat}(q) - 1$, equivalently, by Theorem 3.30, the triple $\left((M^m, \tau); \mathbb{R}^{\mathrm{secat}(q)}\right)$ does not satisfies the BUP.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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